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AN EMPIRICAL STUDY ON THE USE AND TRANSITIONS OF REPRESENTATIONS IN PRIMARY MATH LESSONS

Ildze Čakāne[1], Ilze France[1]

[1] University of Latvia, Latvia

ABSTRACT

The goal of mathematics education is no longer the fast production of correct computations that technology can do more precisely and quicker than a human. An important goal today is to develop the ability to see the use of mathematical concepts as problem-solving tools. To be able to use mathematical ideas creatively one must first understand them. To foster conceptual understanding of mathematics one needs to make connections among concepts. For that, instructional content should incorporate diverse representations of mathematical ideas, and students should be given opportunities to link these representations together. This study aims to explore the use of representations and the connections between them in elementary math classrooms. Participants were 11 teachers from Latvia, Sweden and Norway. The data gathering was self-reflective. Teachers recorded the representations and transitions they used in their mathematics lessons. Beforehand, a consensus on the understanding of five representations (visual, math symbols, language, real-world and manipulatives) was gained via workshops and discussions. Closer understanding of transitions was achieved by collectively watching and analyzing a lesson video recording and agreeing on the transitions seen. The results show that manipulative and real-world situations are the least used in the lessons and the most used transitions were from visual to math symbols and from language to math symbols. While the importance of representation use is generally acknowledged in elementary math education, our study reveals a noticeable gap in the integration of concrete objects and real-life situations in everyday lessons. This raises a question about bridging the gap between research insights and actual teacher practices. Identifying strategies to enhance the incorporation of manipulatives and real-world scenarios in classrooms becomes imperative for fostering a deeper conceptual understanding of mathematics among students.

Keywords: *elementary math, making connections, representations, transitions, manipulatives, real-world situations, teaching for understanding.*

Introduction

Mathematics education aims for students to gain mathematical literacy, to learn to reason mathematically and to be able to use math in diverse real-world situations (Organization for Economic Co-operation and Development [OECD], 2018). However, findings from the OECD's PISA2022 report reveal a concerning gap: on average, only 9% of 15-year-olds globally reach level 5 (in Latvia – 6%, Sweden – 10%, Norway – 7%). Level 5 indicates that a student can develop models of complex situations and work within the model, are able to justify actions, and evaluate the results. (OECD, 2022). It has been shown that student performance in later years is heavily predicted by their experience in primary school and the foundational math skills gained during this time (Duncan et al, 2007). This connection underscores the importance of early education in shaping future outcomes, which brings us to the age group of this research. A fundamental element of math learning for understanding is learning mathematical ideas by experiencing them through multiple representations and transitions between them (Van de Walle et al., 2018). Representations are beneficial for primary students in learning and applying abstract mathematical concepts. These tools are particularly effective when used to facilitate the understanding of new concepts and enhance students' problem-solving abilities. Although there is robust and strong support for students to explore these transitions, "questions about how to develop students' transitioning from one representation to another remain unsolved" (Sokolowski, 2018). To gain a better understanding of how teachers are incorporating the use of different representations, and more importantly the transitions between them for understanding math ideas in their classrooms, a trial empirical research study was conducted that involved teachers recorded their use of representations in a protocol and participated in a reflection session.

Aim of the study: To explore teacher-administered use of representations and their connections in elementary math classrooms.

Research questions:

1. What is the frequency of use of each of the five representations (static pictures, manipulative models, written symbols, real scripts and spoken language) in math lessons in elementary school?
2. What are the most commonly used transitions between representations in math lessons in elementary school?

Theoretical background

Benefits of a use of a variety of representations in math learning

The authors agree with the idea that understanding mathematics is "the ability to represent a mathematical idea in multiple ways and to make connections among different representations" (Cramer & Karnowski, 1995). The idea that math understanding is not an undetectable phenomenon or a mystical feeling, but the ability to translate ideas, concepts and solutions in various ways. This idea is backed by many recent research studies. For example, using diverse mathematical representations such as graphs, equations, and real-world models enables students to see the same information from different perspectives, thus fostering a deeper comprehension of the underlying mathematical concepts (Ozgun-Koca, 1998). Providing varied representations can help students abstract essential mathematical ideas more effectively (Sokolowski, 2018). Younger students who are better at solving problems using manipulatives are more likely to use sophisticated strategies later in school (Siegler, 1993). Linking representations is a component of children's number sense (Sarama & Clements, 2009). For learners to be proficient in learning mathematics, the ability to translate one representation to another is an important skill that needs to be developed (Mainali, 2021). Students who have difficulty translating a concept from one representation to another also have difficulty solving problems and understanding computations (Lesh et al., 2003).

The authors would like to point out a potential misunderstanding: that it is sufficient to use just manipulatives and visual depictions of mathematical ideas for students to gain understanding. Teachers have to explicitly show or provide experiences where students link manipulative or visual representations to the symbolic analog, otherwise, students may have the impression that a mathematical idea they experienced with manipulatives is different from the one they learn when working with symbols. For example, a student shows how a new ten is formed when adding two numbers, but then when he learns the addition algorithm the student does not realize how the blocks on his desk are linked to the numbers in his notebook. This happens because "mathematical relationships do not exist in objects and children do not acquire these relationships through empirical abstraction from objects" (Kamii et al., 2001). The "how" of using various representations in the classroom should be considered not just the "if", because there might be all kinds of situations in the classroom that do not embrace the potential of representations. For example, a teacher giving manipulatives to students so they can use them to solve a problem without explaining or modeling how the manipulatives can help (providing dienes blocks for the first time for calculating three-digit addition). Another example is a teacher showing how to get the right answer for an operation by using visual aids, but not exploring why it works ("jump down in a hundred square when adding tens, jump to the right when adding ones").

In the well known research Visible Learning conducted by John Hattie only a surprisingly average effect size (0.39, where the desired effect size starts from 0.4) for Manipulative materials on math was recognized (Hattie, 2023). This indicates that manipulative use has a potential to accelerate learning but does not reach the potential to considerably accelerate student achievement. The research studied in this meta-study regarded the existence and impact of various representations used in lessons, not the quality of use. This reinforces the importance of the quality of the representation use, the necessity for students to see the interconnections, and to link the representations. Proficient problem solvers instinctively switch to the most convenient representation at any point of the problem-solving process, they are flexible in using relevant representations (Warner et al., 2002).

Representation framework

The authors would like to start this subsection with a citation from Bruner and Kenney's highly influential work "Representation and Mathematics Learning" (1965) where their ideas about bridging the gap between abstract mathematical concepts and student's existing cognitive structures with the help of concrete, pictorial and abstract representations (enactive, iconic, and symbolic stages of representation) are described:

> "With the help of a symbolic notation that remains invariant across transformations in imagery, the learner comes to grasp the formal or abstract properties of the things he is dealing with." (Bruner & Kenney, 1965, p. 56)

After Bruner's idea of learning mathematical ideas by starting with concrete models (manipulatives), then moving to pictorial images and finishing with abstract representations, a more versatile and multidirectional model emerged (see Figure 1). Since Lesh, Post and Behr published their "Representations and Translations among Representations in Mathematics Learning and Problem Solving" (Lesh et al., 1987) it has been the basis and main reference for scholars and practitioners when discussing representation use in math education. The five representations are:

1. evidence-based scripts – real-world situations, the context;
2. manipulative models – physical objects that students can use to explore and understand mathematical concepts;
3. pictures or diagrams – static figural models, internalized "images";
4. spoken language;
5. written symbols, that as well as spoken language involve math specific phrases as well as everyday language.

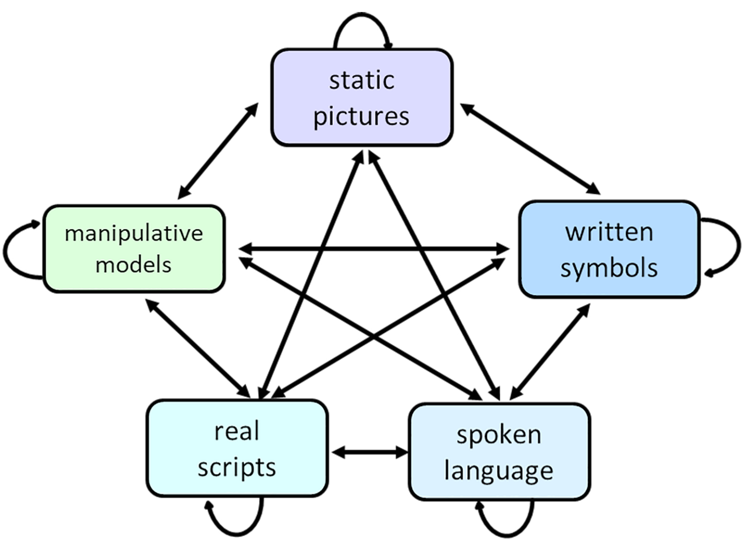

Figure 1 Author visualization for Lesh Translation Model

There are some expanded versions of this model, for example, Johnson in her Mathematical Representational Model (Johnson, 2018) added Technological representation to the basic five representations. She discusses mostly the availability of other representations via technology, not Technological representation as a different form of showing mathematical ideas. The nuisance change and probably why Johnson chose to separate it out is that technology provides opportunities for "moving pictures", also known as virtual manipulatives, such as sliding counters, grouping base ten blocks, adding fractions etc. The use of animated pictorial representations might help to bridge the gap between manipulative models to static pictures. Another example of an expanded version of the Lesh's Translation model is the Web of Representations (Van de Walle, 2018) that shows the ways students can demonstrate their understanding. In this model the pictorial component is divided into three separate ones: create a graph; display data in a table; and draw a diagram. The other representations are directly related to the basic ones: give a context (real-life example); explain meaning in words; illustrate with physical tools; write using symbols.

For this research, we chose to use the original model – Lesh's Translation Model. We opted to forego Johnson's model since usually the technological representation is one of the basic five, just administered via some form of technology. It would also complicate data gathering for teachers since they would have to decide whether a chosen representation is technological or pictorial or other. The Web of Representation is not suitable, since it specifically describes ways a student can show his understanding not build his understanding.

Methodology

As a first step to investigate how teachers are using transitions between representations to help students make meaning of math, a workshop was organized to align understanding of representations of mathematical ideas used in lessons. Teachers discussed each representation and created examples so that all participants in the research was using the same terms for the same ideas (see Figure 2). Teachers decided to use the term "language" for any form spoken or written instead of "oral language" or "verbal" as described in the theoretical literature, and the term "math symbols" for greater clarity instead of "symbolic" or "written symbol" to avoid discrepancies. The term "transitions" was used, not "translations" when referring to connecting one representation to another or another form of the same representation, thus avoiding miscommunication possibilities. This was agreed upon since the discussions occurred in English by non-native speakers and the word translations was confusing.

The sample consisted of 11 primary school teachers (2 from Sweden, 5 from Norway, and 4 from Latvia). This was a convenience sample formed of teachers participating in a lifelong learning project to explore and improve one's teaching. When joining the project teachers agreed that the data they provide will be used in research. Recordings/notes of 107 lessons were made displaying the use of representations. In regards to tracking the use of transitions 7 out of all the teachers made notes about 71 lessons. These activities were proposed for teachers to reflect on their practice from a specific perspective to ensure a purposeful discussion inbetween the participants after the data gathering.

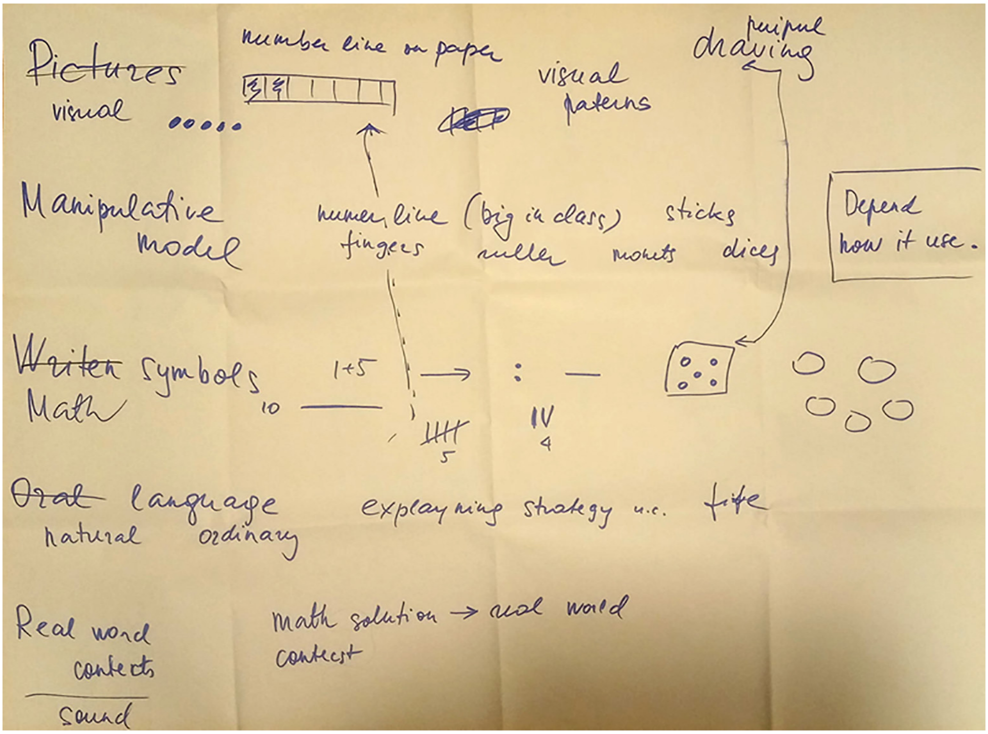

Figure 2 An example of notes of the discussion made by a participant

Date	Visual representation	Math symbols	Language	Real-world situations	Manipulative models	Translations between and within	Clarification, short description	Topic, aim of the lesson/lessons
10.1.2023	x	x	x			**Language→Visual representation** **Visual representation→ Math symbols** **Visual representation→Language**	Reads the description of the situation, depicts it in a schematic drawing (draws the whole, divides it into equal parts, writes down the given value, calculates the value of the basic part, calculates the value of the part). Explains what is given, what needs to be calculated, the solution	Create a shematic drawing to represent the given situation

Figure 3 Example of a teacher's recordings of representations and transitions

Second, teachers were asked to observe a video of a math lesson and record each example of a representation used and, more importantly, what transitions between those representations they noticed. Afterwards, teachers agreed on what was seen regarding the depiction of math ideas in the video.

When common ground was set, teachers were asked to reflect on their own math lessons for one grade for at least two weeks and record what kind of representations they used in each lesson as well as the transitions. An example is shown in Figure 3.

When data was collected a reflection session was organized, where teachers shared their experiences, observations and insights gained while recording their observations. Then the common results were shown to them and further discussion was led in order to pinpoint which results were intuitive and predictable and what surprised or disappointed the participants.

Results

RQ1 The use of representations

The average use of each form of representation was calculated for each teacher (see Table 1). For example, teacher 1 used manipulatives in 67% of his lessons. Most of the teachers used Visual, Math symbols and Language representations more than Real-world and Manipulative models except teachers 5 and 11. Teacher 5 used Real-world models more than Visual representations, but teacher 11 used Manipulatives the most with Visual representations in second place.

The average use of each form of representation was calculated overall (see Figure 4). The most common representations used were Language, Visual and Math symbols, but the least used were Manipulatives and Real-world situations. In 37 of all lessons (35%) neither Real-world situations nor Manipulatives were used.

Table 1 Frequency of representation use individually

Teacher	Visual	Math symbols	Language	Real world	Manipulatives
1	92%	100%	100%	25%	67%
2	100%	100%	100%	60%	60%
3	100%	86%	100%	57%	43%
4	93%	73%	87%	27%	53%
5	67%	89%	100%	78%	33%
6	100%	100%	100%	60%	60%
7	100%	80%	60%	0%	20%
8	71%	100%	100%	43%	43%
9	95%	86%	90%	10%	19%
10	67%	67%	100%	0%	33%
11	75%	58%	67%	50%	91%

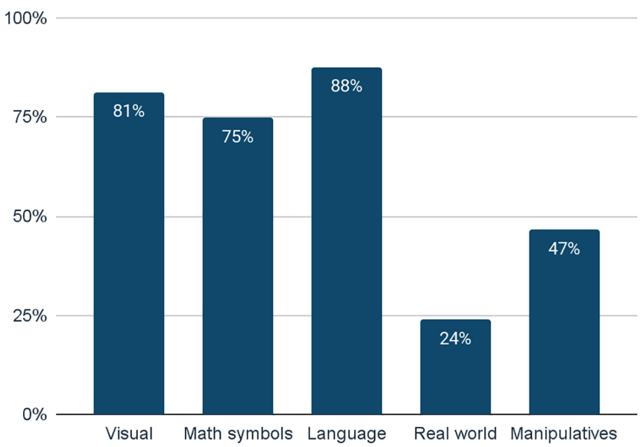

Figure 4 The average use of a representation overall

RQ2 The use of transitions between representations

Similarly, the average use of each transition was calculated for each teacher (see Table 2) and overall (see Figure 5). The following are abbreviations used in the table and figure: Visual representation (V); Math symbol (S); Language (L); Real world (R) and Manipulatives (M). If the teacher noted that in the lesson he used a transition from a visual representation to a symbolic it is denoted followingly: V→S. Only the transitions that at least one teacher recorded are depicted.

The transitions used in classrooms differed greatly from teacher to teacher, and the usages of a transition varied up to 86% (V→S).

The most common transition teachers claimed to be using in their lessons was Visual to Symbolic (49%), followed by Language to Symbolic (42%) and Visual to Language (35%).

Surprisingly the transition Manipulatives to Visual representation was used by only 3 out of 7 teachers in their practice (and two of them very sparsely in 5% and 8% of their lessons), although it is the Bruner suggested learning trajectory.

The authors noticed, that when looking at the directions of the transitions, symbolic representations are more frequently the representation that is at the end of the transition, meaning that it is the representation that describes the math idea previously shown by another representation. Inversely, manipulative models are most frequently the beginning part of a transition, and later described using another representation.

Table 2 The average use of a transition of representations individually

Teacher	V→S	L→S	V→L	M→L	M→S	L→V	M→V	S→S	R→L	S→L	R→V	S→V	S→M	L→M	V→V	S→R	R→S
1	75%	25%	8%	58%	58%	0%	58%	0%	17%	8%	8%	0%	0%	0%	0%	0%	0%
2	20%	80%	40%	40%	20%	0%	0%	0%	20%	0%	20%	0%	0%	20%	0%	20%	0%
7	60%	20%	40%	0%	20%	0%	0%	20%	0%	0%	0%	0%	20%	0%	0%	0%	0%
8	86%	86%	0%	14%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%	0%
9	67%	10%	71%	19%	19%	52%	5%	38%	10%	14%	0%	19%	0%	0%	5%	0%	5%
10	33%	67%	67%	33%	22%	22%	0%	0%	0%	22%	0%	22%	11%	11%	0%	0%	0%
11	0%	8%	17%	8%	0%	0%	8%	0%	8%	8%	17%	0%	0%	0%	17%	0%	8%

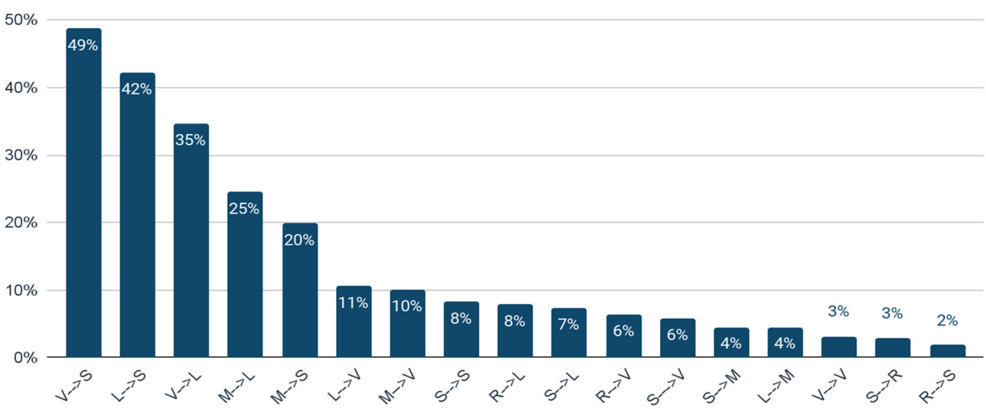

Figure 5 The average use of a transition of representations overall

Insights from teacher reflections

At the beginning of the research, teachers were skeptical, and not keen to add more duties to their day, but after the reflective work and gathering data on their practice, all of them agreed they gained some form of benefit. The necessity to reflect on their practice led to better awareness, and understanding of their own practice especially from the perspective of representation use. The experience resulted with a determination by many participants to change their practice to add more of a specific representation in their teaching. For precise teacher insights see Table 3.

Table 3 Teacher reflections after recording their use of representations

Teacher	Excerpt from transcript
1	I saw that mostly it was always one way in my lessons: models or concrete objects to symbols.
3	It gave me a more clear picture of how my lessons are.
4	What we [talking about himself and his colleague, teacher 5] understood, was that we do a lot of representations, but we were not aware of that. That has been a learning point, to talk about with colleagues, I hope, that after this tracking experience my lessons will be a little bit better.
6	I am more aware of how I teach, I use more manipulative models now, because I noticed, that I didn't do it enough.
11	I noticed that I use a lot of pictures, which is good, but my takeaway from this is that I need to incorporate more real-world situations.

Discussion and conclusions

Noteworthy are the results, that Visual representations were one of the three most frequently used representations alongside Language and Math symbols, which means that their students experienced visual depictions of math ideas in lessons, not just abstract forms of math. Not surprisingly the least used were Manipulatives (47%) and Real-world situations (24%). These findings pertain to existing research on the subject, arguing that although teachers have an understanding of the importance of these representations, their practice is not consistent with it (Spillane & Zeuli, 1999; Moyer, 2001; Boaler, 2002). It is hard to argue if it is enough that students had the opportunity to use concrete models in almost half of the lessons, but we should not forget the importance of experiencing math ideas concretely, especially in early education. The most concerning is the fact that in only a quarter of lessons, students had the opportunity to connect their abstract math knowledge to everyday life, which is the ultimate goal of math. Here the authors see the gap between scholarly research and practitioners, meaning that the theoretically desirable practice is not fully brought to everyday classrooms.

Another noticeable aspect based on the gathered data is that students have very different experiences learning math in each classroom –this notion is consistent with existing research (e.g., Hiebert & Grouws, 2007; Čakāne et al., 2024). For example, in some (2 participating teachers) classrooms there was no records of real-world representations, but in others (4 teachers) in more than half of the lessons. Another illuminating fact is that teachers have their "favorite" transitions that they use more frequently and some they don't use at all (or don't recognize them if they do) and these differed greatly from teacher to teacher. This raises a question of whether all students have equal opportunities for qualitative education to reach their potential. Here the authors would like to bring to attention the lack of research and examples of good practice of illustrated transitions and the benefits of each or the most suitable ones for specific mathematical ideas that might help to bring theory to practice. Guidelines for teachers to follow on how to make connections between different representations would be beneficial.

Teachers mostly chose to use transitions that end with Language or Symbolic (V→S, L→S, V→L, M→), and very rarely ones that end with Visual or Manipulative (R→V, S→V, S→M, L→M). This signals that students mostly experience translations from more concrete representations to more abstract ones. That brought the authors to think about the Web of Representations and the importance for students to be able to show their understanding diversely where inverse transitions are highly useful. A student's ability to show an equation with manipulatives or provide real-life context can identify a more profound understanding than calculating the right answer. Whether the low frequency of transitions ending with more concrete representations was sufficiently used in the lessons is hard to argue, but the authors tend to think that those should be incorporated more often. To the authors' knowledge, there is no other research to the present day that looks specifically on the use of transitions of representations in elementary math classrooms.

Alongside the main aim of this research, it was noticeable that it is not a part of participating teacher everyday practice to reflect on their teaching from a specific perspective. Teachers' beliefs about their practices varied from what the data showed, and they came to conclusions about what they should change by themselves only by reflecting on their teaching and recording representations they used. The reflective nature of data gathering was beneficial for teacher professional awareness and illuminated the positives and negatives of their practice and could be incorporated into teacher professional development models as a valuable component.

Limitations

The participants' understanding of each representation might have been different regardless of the discussion and examples provided. The perception of representations or transitions the teachers recorded as being used in their lesson are also subject to interpretation.

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